Approximate Large Deflection Analysis of Thin Rectangular Plates under Distributed Lateral Line Load

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ABSTRACT
Research on large deflection of thin rectangular plates to date has focused on plates under uniformly distributed load. There is need to extend the theory to other forms of loads and to get simpler approximate solutions that are easier to use. An approximate analysis of large-deflection of thin rectangular plates under distributed lateral line-load is presented in this paper. The analysis is based on solving Von Kármán equations. The load, deflection and stress are represented by single trigonometric series in the x-direction and a truncated cosine function in the y-direction is assumed. The functions are substituted into the Von Kármán equations to get third degree polynomials describing relationships between load and deflection coefficients. The polynomials are solved using scientific workplace function *polynomialsroots*, to get defections caused by different loads. Results are compared against the exact solution and it is seen that the two results show similarity in the trends of relationships between loads and deflections, and deflections and number of coefficients, but deflections obtained using these approximations are higher than of the exact method for the same load. The proposed approximate method is seen to be simpler and could be adopted where accuracy is not very critical.

Keywords: Approximate, Exact, Large deflection; Line-load; Stress; Von Kármán equations.

1.0 INTRODUCTION

Owing to the complexity of exact solutions for large deflection analysis, several approximate solutions for uniformly-loaded plates of simple, regular shapes have been suggested, notably by Timoshenko, but they lead to significant loss in accuracy (Timoshenko and Woinwsky-Krieger, 1959). Wang and El-Sheikh (2005), suggested an approximate mathematical solution for plates under uniformly distributed load acting laterally. Their solution is based on solving Von Kármán equations (equations 1a, 1b), but they used the *Microsoft excel solver* for solving the final load-deflection equations. Radoslav (2000) suggested an approximate solution for plate strips under tension in the longitudinal direction. The tension is in the form of in-plane forces pulling the plate in opposite directions.

$$\frac{\partial^4 F}{\partial x^4}+2\frac{\partial^4 F}{\partial x^2 \partial y^2}+\frac{\partial^4 F}{\partial y^4}=E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{1a}$$

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D}+\frac{t}{D}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}+\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) \tag{1b}$$

w is the deflection of the plate, E is Young's modulus, t is the plate thickness, q is the distributed load, and D is the plate flexural rigidity such that:

$D = \frac{Et^3}{12(1-v^2)}$, v is Poisson's constant, and F is Airy stress function such that $\frac{\partial^2 F}{\partial y^2} = \sigma_x, \frac{\partial^2 F}{\partial x^2} = \sigma_y, \frac{\partial^2 F}{\partial x \partial y} = -\tau_{xy}$. σ_x, σ_y are the normal stresses in x and y directions, τ_{xy} is the shear stress in x-y plane.

$$F = -\frac{\overline{p}_x y^2}{2} - \frac{\overline{p}_y x^2}{2} + \sum_{m=0,}^{\infty}\sum_{n=0,}^{\infty} f_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad (1c)$$

$\overline{p}$ denotes in-plane loads. $\overline{p}$ =0 for lateral loads.

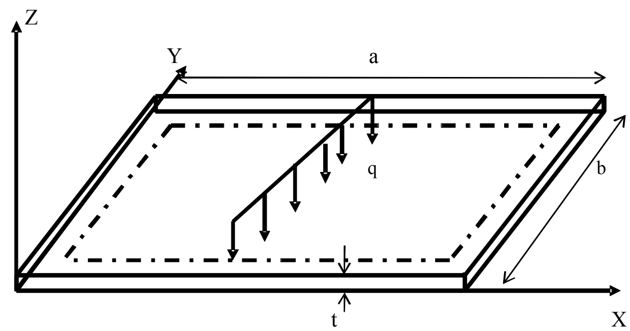

Figure 1: Typical simply supported plate under distributed line load, (adapted by authors from literature)

2.0 METHODOLOGY

This paper presents the findings of a study that analyzed the behaviour of thin rectangular plates subjected to uniformly distributed lateral line-loads using an approximate method based on assumption of a truncated cosine function (Okodi, 2010). The specific objectives were to derive and solve approximate load-deflection relationships for thin rectangular plates with simple supports, subjected to distributed lateral line loads and to compare the outcome with results obtained using the exact method. The following steps were followed to achieve the aforementioned objectives:

i) Derivation of mathematical relationships between load and deflection coefficients for plates under lateral line-loads,
ii) Solution of the derived relationships to obtain the deflections caused by various loads using proprietary software (Scientific work place),
iii) Graphical presentation of results.

The research was limited to thin rectangular plates subjected to distributed lateral line-loads with simply supported edges.

Consider the simply supported rectangular plate shown in Figure 2.

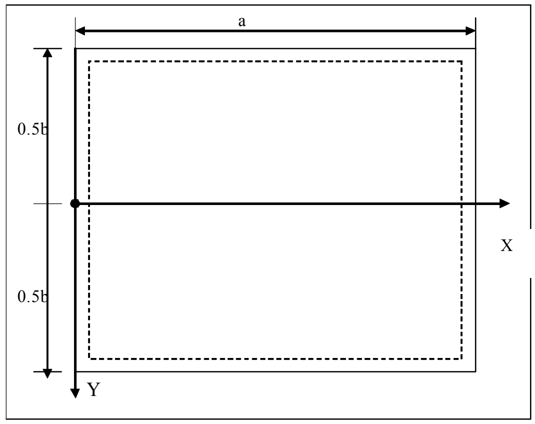

Figure 2: Simply supported plate, with Levy's axes
(Adopted by authors from literature)

For a load q acting along the x-axis, load intensity is given by

$$q(x) = \frac{q}{a} \tag{2a}$$

Load is also represented by Fourier series in x direction as follows

$$q(x) = \sum_{m=1}^{\infty} q_m \sin\frac{m\pi x}{a} \tag{2b}$$

By integration, it can be shown that coefficients q_m are defined by the equations below,

$q_m = \frac{4q}{m\pi a}, m = 1,3,5,.....\infty$

Substituting for q_m in the load series yields the expression below

$$q(x) = \frac{4q}{\pi a}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{m\pi x}{a} \tag{2c}$$

The general deflected shape would be as follows:

$$w(x) = \sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a} \tag{3a}$$

where Y_m is the coefficient of deflection and is a function of y only. It must be such that it satisfies the boundary conditions. That is to say, $y = \pm\frac{b}{2}, w = 0, and\ , \frac{\partial^2 w}{\partial y^2} = 0$.

The cosine function is considered in this paper and a maximum of five terms are used in determining the deflected shape of the plate. Choosing very many terms significantly increases the volume of work to be done although accuracy is improved. Only odd values of *m* satisfy the boundary conditions and therefore even values are ignored.

$$Y_m = w_m \cos\frac{m\pi y}{b} \tag{3b}$$

So,

$$w(x) = \sum_{m=1}^{\infty} w_m \cos\frac{m\pi y}{b}\sin\frac{m\pi x}{a}, \tag{3c}$$

Leading to the following deflected shape equations

$$w(x) = w_1 \cos \frac{\pi y}{b} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for one deflection term}$$

$$w(x) = \left(w_1 \cos \frac{\pi y}{b} + w_3 \cos \frac{3\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for two deflection terms}$$

$$w(x) = \left(w_1 \cos \frac{\pi y}{b} + w_3 \cos \frac{3\pi y}{b} + w_5 \cos \frac{5\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for three deflection terms}$$

$$w(x) = \left(w_1 \cos \frac{\pi y}{b} + w_3 \cos \frac{3\pi y}{b} + w_5 \cos \frac{5\pi y}{b} + w_7 \cos \frac{7\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for four deflection terms}$$

$$w(x) = \left(w_1 \cos \frac{\pi y}{b} + w_3 \cos \frac{3\pi y}{b} + w_5 \cos \frac{5\pi y}{b} + w_7 \cos \frac{7\pi y}{b} + w_9 \cos \frac{9\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for five deflection terms}$$

Expressions for Airy stress are similarly derived, and are shown below.

$$F(x) = f_1 \cos \frac{\pi y}{b} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for one stress terms}$$

$$F(x) = \left(f_1 \cos \frac{\pi y}{b} + f_3 \cos \frac{3\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for two stress terms}$$

$$F(x) = \left(f_1 \cos \frac{\pi y}{b} + f_3 \cos \frac{3\pi y}{b} + f_5 \cos \frac{5\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for three stress terms}$$

$$F(x) = \left(f_1 \cos \frac{\pi y}{b} + f_3 \cos \frac{3\pi y}{b} + f_5 \cos \frac{5\pi y}{b} + f_7 \cos \frac{7\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for four stress terms}$$

$$F(x) = \left(f_1 \cos \frac{\pi y}{b} + f_3 \cos \frac{3\pi y}{b} + f_5 \cos \frac{5\pi y}{b} + f_7 \cos \frac{7\pi y}{b} + f_9 \cos \frac{9\pi y}{b} \right) \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \qquad \text{for five stress terms}$$

Differentiation and substitution of the expressions for deflection and stress function into Von Kármán equations result in equations with *f* and *w* as unknowns. By taking only the first terms for example, the following equations (4a and 4b) are obtained.

$$f_1 \cos \frac{\pi y}{b} \left[\frac{1}{a^4} \sum_{m=1}^{\infty} m^4 \sin \frac{m\pi x}{a} \quad \frac{2}{a^2 b^2} \sum_{m=1}^{\infty} m^2 \sin \frac{m\pi x}{a} \quad \frac{1}{b^4} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \right] =$$
$$\frac{E w_1^2}{a^2 b^2} \left[\sin^2 \frac{\pi y}{b} \left(\sum_{m=1}^{\infty} m \cos \frac{m\pi x}{a} \right)^2 - \cos^2 \frac{\pi y}{b} \sum_{m=1}^{\infty} m^2 \sin \frac{m\pi x}{a} * \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \right] \tag{4a}$$

$$\pi^4 w_1 \cos \frac{\pi y}{b} \left[\frac{1}{a^4} \sum_{m=1}^{\infty} m^4 \sin \frac{m\pi x}{a} + \frac{2}{a^2 b^2} \sum_{m=1}^{\infty} m^2 \sin \frac{m\pi x}{a} + \frac{1}{b^4} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \right] = \frac{4q}{\pi a D} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{m\pi x}{a} +$$
$$2 \frac{\pi^4}{a^2 b^2} f_1 w_1 \frac{t}{D} \left[\cos^2 \frac{\pi y}{b} \sum_{m=1}^{\infty} m^2 \sin \frac{m\pi x}{a} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \quad \sin^2 \frac{\pi y}{b} \left(\sum_{m=1}^{\infty} m \cos \frac{m\pi x}{a} \right)^2 \right] \tag{4b}$$

These are two expressions in two unknowns (f_1, w_1). Thus we are able to determine the unknowns at any location *(x, y)* on a plate of known dimensions. To demonstrate the solution procedure, consider the case of deflection at the centre of a square plate. Plate dimensions $(a = b)$, and location (x, y) is $\left(\frac{a}{2}, 0\right)$.

For one term $(m = 1)$, $\frac{\pi^4}{a^4} w_1 [1 + 2 + 1] = \frac{4q}{\pi a D} + \frac{2\pi^4}{a^4} f_1 w_1 \frac{t}{D} [1 * 1]$

Substitute for f_1: $4 \frac{\pi^4}{a^4} w_1 = \frac{4q}{\pi a D} - \frac{\pi^4}{2a^4} w_1^3 \frac{Et}{D}$

On further simplification, the above expression becomes,

$$0.5w_1^3 + \frac{4D}{Et}w_1 - \frac{4qa^3}{\pi^5 Et} = 0 \qquad (5)$$

This is the load-deflection relationship used to get the deflection coefficients for any load. Consider one deflection coefficient; the deflection at the centre is the same as the coefficients. By following the same procedure, relationships for 2, 3, 4 and 5 deflection terms have been derived for simply supported plates. These expressions have been solved by computer software for 1m*1m steel plate of 1mm thickness.

3.0 RESULTS

From Table 1, the deflections are seen to increase with load as would be expected. This is similar to results of the exact method. Table 1 results have been plotted to show the trend of deflection with number of deflection terms (Figure 3), and to show the trend of deflection with load (Figure 4). An additional plot (Figure 5) of load against deflection has been made to compare the results of the exact and approximate methods for 1mm thick plate.

Table 1: Showing approximate deflections for various numbers of terms

Load	Deflection w(mm) for different numbers of terms				
(N)	1 term	2 terms	3 terms	4 terms	5 terms
0	0	0	0	0	0
1	4.9849	2.6284	3.2365	3.2343	3.2361
3	7.2224	3.6420	4.6090	4.5273	4.6311
5	8.5746	4.2696	5.4612	5.4876	5.4922
7	9.5987	4.7518	6.1093	6.1419	6.1473
9	10.4420	5.1509	6.6437	6.6812	6.6873

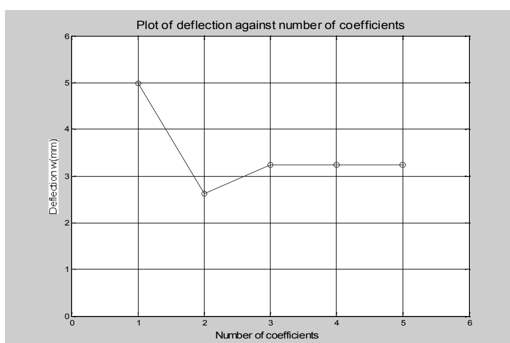

Figure 3: Plot of approximate deflection against number of terms

The plot of deflection against number of coefficients or terms (using results of 1N load) in Figure 3 above shows the following:

a) The value of deflection is highest for one term (47.28% higher than the lowest),
b) The value is lowest at two terms (47.28% lower than value at one term),
c) The value at three terms is 35.07% lower than the highest but 12.21% higher than the value at two terms, which is the lowest,
d) The value at four terms is 35.12% lower than the highest but 0.05% lower than the value at three terms that precedes it.
e) The value at five terms is 35.08% lower than the highest value but 0.04% higher than the value at four terms.

The plot in figure 3 further indicates that the deflections fall steeply when the number of terms is increased from one to two. However the deflection there-on generally reduces with increasing number of terms albeit with extremely low gradient. The fall in the value of deflection when the number of terms is increased from one to two, the rise there-on and the tendency to stagnate on a single value (approx 3.235mm in this case) is in-keeping with the results obtained by other researchers who used exact method to study uniformly loaded plates, notably Wang and El-Sheikh (2005). This shows that convergence is faster for the approximation chosen. However the deflections are very high and the difference between the deflections obtained using one term is extremely high compared to deflections obtained using higher terms. This is at variance with results of the exact method and is likely to be caused by the assumed mathematical relationship between coefficients of deflection and coefficients of Airy stress.

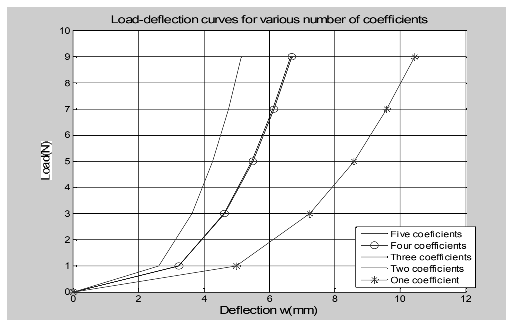

Figure 4: Plot of load against approximate deflections

The plot of load against deflection (Figure 4) indicates that the deflections increase non-linearly with increasing load, which is in-keeping with the theory of large deflection. However there is a marked variation in values as the number of deflection coefficients (terms) is increased. As explained above, this large variance is likely to be caused by the assumed mathematical relationship between coefficients of deflection and coefficients of Airy stress. Figure 4 also shows that the plots for one term and two terms are separate lines with clear distances apart. The plots for three, four and five terms are close together (banded together into a bunch). This implies that one coefficient (term) over-estimates the deflection, two coefficients under-estimates the deflection, while the deflection converges by the third coefficient (term) to a value, which varies with load.

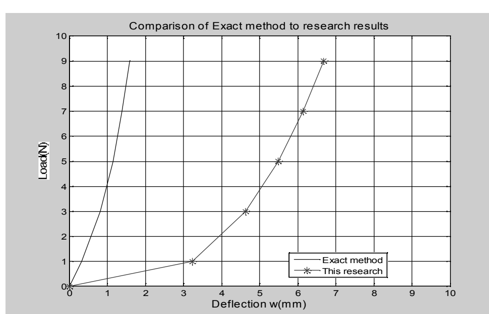

Figure 5: Plot of load against deflection for results of exact and approximate methods

The plot in Figure 5, of load against deflection (for results obtained using one coefficients with both exact and approximate methods) shows that both methods maintain a non-linear relationship between load and deflection of thin plates, which allows the theory of large deflection to hold.