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Option Pricing: Lattice Models Revisited

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Abstract

In this article the topic of option pricing using lattices is re-examined. A moment matching technique and the method of finite differences are used to develop a parametrization for trinomial and binomial lattices. In addition, the CRR model is revisited for which alternative up and down factors are provided.

Introduction

As a result of the famous paper by Black and Scholes (1973), the last three decades have recorded a tremendous success in the valuation of options ranging from the simple vanilla options to more complex options. However, as closed form solutions are rare (especially for exotic options), numerical solutions are used as the best substitutes.

One such method is based on approximating the underlying continuous process with an appropriate discrete process. This method is commonly referred to as the moment matching technique. In this class are the binomial models of Cox et al. (1979)(in short CRR), Rendleman and Bartter (1979), Trigeorgis (1991) and the ABMC and ABMD models of Jabbour et al. (2001). The parameters for the binomial models of Cox et al.; Trigeorgis ; Rendleman and Bartter were determined by equating the mean and variance of the log-transformed underlying distribution to the mean and variance respectively of the binomial approximating distribution. The ABMC model of Jabbour et al. was obtained by matching the mean and standard deviation of the (non-transformed) underlying lognormal distribution with the mean and standard deviation respectively of the approximating discrete distribution, while the ABMD was obtained by matching the mean and variance of the proportionate change in the value of the underlying asset.

The moment matching technique was extended to trinomial lattice by Boyle (1986, 1988). In Boyle's method, jump probabilities were obtained by matching the mean and variance of the approximating distribution with the mean and variance respectively of the underlying lognormal distribution. To ensure nonnegative probabilities, a constrained parameter for the jump parameters was introduced. Boyle also derived a five-jump model to approximate a bi-variate lognormal distribution. Extending this model to three or more state variables was difficult because of the non-negativity requirement of the jump probabilities. By using an alternative process, Boyle et al. (1989) were able to overcome this problem. Kamrad and Ritchken (1991) used a moment matching technique to match the mean and variance of the log transformed underlying distribution. Their model extends well to any number of state variables and includes many existing models as special cases. Jabbour et al. (2005) provided a step by step moment technique for developing an $n-$order multinomial lattice parametrization for a single-state option pricing model. Chen et al. (2002) discussed a log-transformed trinomial approach to option pricing and found out that various numerical procedures in the option pricing literature are embedded in their approach with choices of different parameters. They compared the efficiency of numerous schemes and were able to conclude that the equal probability trinomial specification of He (1990) and Tian (1993), and the sharpened trinomial specification of Omberg (1988) outperformed others.

What is common to all these models is that a moment matching technique is used to develop a discrete process consistent with the underlying stochastic process in the sense that for each time step, the mean and variance of the underlying continuous distribution is matched with the mean and variance of the discrete distribution respectively. The efficiency of these models depend on how good these approximations are. A more efficient scheme takes less computational time than a less efficient one.

An alternative method is to approximate the Black and Scholes partial differential equation with an appropriate difference scheme. In this class are the well known finite difference schemes of Brennan and Schwartz (1977, 1978).

In this article two tasks are carried out. The first task employs the two methods mentioned above to develop a trinomial approximating model. Finite differences are used to deduce probabilities for the jump approximating process and with these probabilities, a moment matching technique is then used to determine jump amplitudes that are consistent with the underlying distribution. This approach gives a more general parametrization of the trinomial jump process from which several models can be deduced by making some simplifying assumptions. In particular, binomial models arise by setting the middle jump probability equal to zero. As a result another parametrization of the two jump process is given.

The second task is to re-examine the well known binomial model of Cox et al. (1979). It is well known (Trigeorgis, 1991, p.319) (Jabbour et al., 2001, p.992) that the CRR model can give rise to negative probabilities. Moreover, the model matches the mean of the underlying distribution but the variance is only matched in the limit as the size of the time steps tends to zero. A correction of this deficiency is given. The resulting models offer some flexibility in the specification of jump amplitudes.

Trinomial Lattices

Let $F(t, S_t)$ be the time t price of a contingent claim whose pay off at a fixed terminal time T is of the form $\Phi(S_T)$. Under the assumptions of the Black-Scholes model, F satisfies the following partial differential equation:

$$\begin{cases} \frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 F}{\partial S^2} - rF = 0, \quad 0 \le t \le T \\ F(T, S) = \Phi(S), \end{cases} \tag{1}$$

where the process S is defined by (A1) and r is the deterministic short rate of interest.

Using the transformation (A2) and Ito's Lemma, (1) can be reduced to the following PDE with constant coefficients:

$$\begin{cases} \frac{\partial F}{\partial t} + (r - \frac{\sigma^2}{2})\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} - rF = 0, \quad 0 \le t \le T \\ F(T, x) = \Phi(e^x) \quad . \end{cases} \tag{2}$$

Let us form a discretization of (2). With T as the fixed time to maturity and N the number of discrete time points, let Δx and $\Delta t = \frac{T}{N}$ be the mesh sizes. If $x_j = j\Delta x, t_n = n\Delta t,\ 0 \le n \le N,\ j \in \mathbb{Z}$, then $F_j^n = F(t_n, x_j) = F(n\Delta t, j\Delta x)$ is the value of the numerical approximation at $(n\Delta t, j\Delta x)$.

Taking the backward difference discretization for time and the second order discretization for space, yields the following difference scheme:

$$\frac{F_j^{n+1}-F_j^n}{\Delta t}+\left(r-\frac{\sigma^2}{2}\right)\frac{F_{j+1}^{n+1}-F_{j-1}^{n+1}}{2\Delta x}+\frac{\sigma^2}{2}\frac{F_{j-1}^{n+1}+F_{j+1}^{n+1}-2F_j^{n+1}}{\Delta x^2}-rF_j^n=0,$$
$$\text{for } j\geq 1,\ 0\leq n\leq N-1, \tag{3}$$

from which one can deduce that,

$$\begin{aligned} F_j^n &= \frac{1}{1+r\Delta t}\left\{\left(\frac{\sigma^2\Delta t}{2\Delta x^2}+\frac{\Delta t(r-\frac{\sigma^2}{2})}{2\Delta x}\right)F_{j+1}^{n+1}+\left(1-\frac{\sigma^2\Delta t}{\Delta x^2}\right)F_j^{n+1}\right\} \\ &+\frac{1}{1+r\Delta t}\left\{\frac{\sigma^2\Delta t}{2\Delta x^2}-\frac{\Delta t(r-\frac{\sigma^2}{2})}{2\Delta x}\right\}F_{j-1}^{n+1},\ \ j\geq 1,\ 0\leq n\leq N-1. \end{aligned} \tag{4}$$

Equation (4) can be written as

$$F_j^n=\frac{1}{1+r\Delta t}\left(q_1F_{j+1}^{n+1}+q_2F_j^{n+1}+q_3F_{j-1}^{n+1}\right),\ j\geq 1,\ 0\leq n\leq N-1;$$

where

$$q_{1,3} = \frac{\sigma^2\Delta t}{2\Delta x^2}\pm\frac{\Delta t(r-\frac{\sigma^2}{2})}{2\Delta x},\ q_2=1-q_1-q_3. \tag{5}$$

With the condition

$$\Delta x\geq \max\left\{\sigma\sqrt{\Delta t},|\mu|\,\Delta t\right\},\mu=r-\frac{\sigma^2}{2}, \tag{6}$$

one can interpret q_1,q_2 and q_3 as probabilities that the stock will jump to the next random value at the end of the current period. In other words, if the current stock price is S_t, then, over the single period $(t,t+\Delta t)$, the stock can jump to a_1S_t with probability q_1, to a_2S_t with probability q_2 and to a_3S_t with probability q_3 where $a_1>a_2>a_3$ and $a_1>1+\widehat{r}>a_3$. Here $\widehat{r}$ is the single period risk free rate.

A moment matching technique is then used to determine the jump amplitudes a_1,a_2 and a_3 which are consistent with the underlying continuous distribution.

As in (A2), the log-transformed equivalent of the discrete approximating process is used. Thus, if $b_j=\ln(a_j), j=1,2,3$ then, the random variable Y defined as

$$Y = \begin{cases} b_1 \text{ with probability } q_1 \\ b_2 \text{ with probability } q_2 \\ b_3 \text{ with probability } q_3 \end{cases}$$

is the discrete distribution used to approximate the continuous underlying normal distribution in (A3).

Equating the mean and variance of the discrete moments with the respective mean and variance of the continuous distribution[1] yields the following system of two equations in three unknowns.

$$q_1 b_1 + q_2 b_2 + q_3 b_3 = \mu \Delta t, \tag{7}$$

$$q_1 b_1^2 + q_2 b_2^2 + q_3 b_3^2 = \sigma^2 \Delta t + (\mu \Delta t)^2. \tag{8}$$

The solution to (7) and (8) in terms of b_2 is

$$b_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

$$b_3 = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

where

$$A = \frac{1}{2\theta^4} - \frac{\mu\sqrt{\Delta t}}{2\theta^3 \sigma},$$

$$B = b_2 \left(\frac{\mu\sqrt{\Delta t}}{\theta\sigma} + \frac{1}{\theta^4} - \frac{1}{\theta^2} - \frac{\mu\sqrt{\Delta t}}{\sigma\theta^3} \right) + \left(\frac{\mu^2\sqrt{\Delta t^3}}{\theta\sigma} - \frac{\mu\Delta t}{\theta^2} \right),$$

$$C = \left(1 - \frac{1}{2\theta^2} + \frac{1}{2\theta^4} + \frac{\mu\sqrt{\Delta t}}{2\theta\sigma} - \frac{\mu\sqrt{\Delta t}}{2\theta^3\sigma} \right) b_2^2 - 2\mu\Delta t(1 - \frac{1}{\theta^2}) b_2$$
$$+ (\mu\Delta t)^2 \left(1 - \frac{1}{2\theta^2} - \frac{\mu\sqrt{\Delta t}}{2\theta\sigma} \right) - \sigma^2 \Delta t \left(\frac{1}{2\theta^2} + \frac{\mu\sqrt{\Delta t}}{2\theta\sigma} \right),$$

$$\theta = \frac{\Delta x}{\sigma\sqrt{\Delta t}}.$$

With the jump amplitudes $\exp(b_j)$ and the corresponding jump probabilities $q_j, j = 1, 2, 3$, the trinomial model is said to be calibrated. The free variables b_2 and θ provide an infinite number of ways in which a three jump process can be calibrated[2]. However, for purposes of computational efficiency, some simplifying restrictions are necessary. Three such restrictions are discussed.

(a) The restriction $p_2 = 0$ reduces the trinomial model to the binomial case. This case will be investigated a little further in Section .

(b) The restriction[3] $b_2 = b_1 + b_3 = 0$ yields

$$b_1 = \sigma\sqrt{\frac{\Delta t}{1 - q_2}} \text{ and } b_3 = -\sigma\sqrt{\frac{\Delta t}{1 - q_2}}. \tag{9}$$

The resulting model is that of Kamrad and Ritchken (1991).

[1] The mean and variance of the continuous distribution are given in the appendix.
[2] θ is a free variable in the limits of condition 6.
[3] For consistency in regard to equation (8), powers of Δt higher than one are neglected.

Let us remember that in the limits of condition (6) there still exists an infinite number of ways to choose q_2. Indeed, Kamrad and Ritchken (1991) computed the difference between the true Black and Scholes value and computed European call prices using a trinomial lattice and found out that for values of q_2 close to one-third, the errors were very small. Horasanli (2007) compares the speed and rate of convergence of the trinomial model of Kamrad and Ritchken (1991) (with $q_2 = 1/3$) with the binomial model of Cox et al. and confirms that the rate of convergence for trinomial models is higher than in binomial models, but the former models are computationally more expensive because they require more computer memory.[4]

When the jump amplitudes in (9) are plugged back back into (8), the resulting value is $\sigma^2\Delta t$ instead of $\sigma^2\Delta t + (\mu\Delta t)^2$. This is the drawback of this approximating process and was noted in the work of Brennan and Schwartz (1978).

Yet another specification would be the following:

$$b_1 = \sigma\alpha\sqrt{\frac{\Delta t}{1-q_2}}, \ \alpha = \sqrt{1+\left(\frac{\mu}{\sigma}\right)^2\Delta t} \text{ and } b_3 = -\sigma\alpha\sqrt{\frac{\Delta t}{1-q_2}}. \tag{10}$$

Specification (10) gives the correct variance but with a higher mean.
As a remark, Kamrad and Ritchken wrote that any choice of $\lambda = \frac{1}{\sqrt{1-q_2}} \geq 1$ makes $q_1 > 0, q_2 > 0$ and $q_3 > 0$. This is not always the case though. For example, if $r = 4\%, \sigma = 6\%, \lambda = \sqrt{5}$ and $\Delta t = \frac{1}{5}$, then q_3 will be negative. Thus, for a given r and σ; Δx and Δt must be chosen in such a way that condition (6) is not violated.

(c) A relatively more general restriction is to have $b_1 + b_3 = 2b_2$ with b_2 not necessarily equal to zero as in (b) above. In this case, we get the following jump amplitudes:

$$b_{1,3} = \mu\Delta t\left(1 - \frac{\sigma}{\sqrt{\sigma^2 - \mu^2\Delta t}}\right) \pm \frac{\sigma^2\sqrt{\Delta t}}{\sqrt{(\sigma^2 - \mu^2\Delta t)(1-q_2)}}, \tag{11}$$

$$b_2 = \mu\Delta t\left(1 - \frac{\sigma}{\sqrt{\sigma^2 - \mu^2\Delta t}}\right). \tag{12}$$

The jump parameters in (11)-(12) were discussed in the work of Chen et al. (2002).

Binomial Models

Letting $\frac{\sigma^2\Delta t}{\Delta x^2} = 1$ in (5) gives

$$q = q_1 = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t} \text{ and } q_3 = \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta t}. \tag{13}$$

These are the jump probabilities for the binomial model. The corresponding jump amplitudes are derived from the following equations:

$$qb_1 + (1-q)b_3 = \mu\Delta t,$$

[4]For N iterations, the trinomial model will require $\frac{N(N+1)}{2}$ more than a binomial model.

$$q(1-q)(b_1-b_3)^2 = \sigma^2\Delta t.$$

In other words,

$$b_{1,3} \quad = \quad \mu\Delta t \pm \sigma\sqrt{\Delta t}\left(\sqrt{\frac{\sigma \mp \mu\sqrt{\Delta t}}{\sigma \pm \mu\sqrt{\Delta t}}}\right).$$

Hence, the following binomial approximating model to the continuous distribution given in (A4).

$$S_{t+\Delta t} \quad = \quad \begin{cases} S_t e^{b_1} \text{ with risk neutral probability } & q, \\ S_t e^{b_3} \text{ with risk neutral probability } & 1-q. \end{cases} \tag{14}$$

Below, model (14) will be referred to as the FDMM binomial model.

Some specifications that are already known can be deduced as special cases of FDMM. For example, in (13), if $\Delta t \to 0$, Then, $q \to \frac{1}{2}$. Therefore,

$$S_{t+\Delta t} \quad = \quad \begin{cases} S_t e^{\mu\Delta t+\sigma\sqrt{\Delta t}} \text{ with risk neutral probability } & 0.5, \\ S_t e^{\mu\Delta t-\sigma\sqrt{\Delta t}} \text{ with risk neutral probability } & 0.5. \end{cases} \tag{15}$$

The binomial model (15) was discussed by Jarrow and Rudd (1983) and Chen et al. (2002).

CRR Model Revisited

Assume that $b_1 + b_3 = 0$ and solve equations (14) and $qb_1^2 + (1-q)b_3^2 = \sigma^2\Delta t$ instead of (14), to get the following specifications.

$$S_{t+\Delta t} \quad = \quad \begin{cases} S_t e^{\sigma\sqrt{\Delta t}} \text{ with risk neutral probability } & \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}, \\ S_t e^{-\sigma\sqrt{\Delta t}} \text{ with risk neutral probability } & \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta t}. \end{cases} \tag{16}$$

This model was discussed by Cox et al. (1979). It can also be deduced from the trinomial model of Kamrad and Ritchken (1991) by setting the probability of the middle jump equal to zero.

Using a replicating portfolio technique and the no arbitrage assumption, Cox et al. (1979) also derived the following binomial model:

$$S_{t+\Delta t} \quad = \quad \begin{cases} S_t e^{\sigma\sqrt{\Delta t}} \text{ with risk neutral probability } & q = \frac{e^{r\Delta t}-d}{u-d}, \\ S_t e^{-\sigma\sqrt{\Delta t}} \text{ with risk neutral probability } & 1-q = \frac{u-e^{r\Delta t}}{u-d}. \end{cases} \tag{17}$$

In fact, it[5] is generally understood that the meaning of the CRR model is the same as (17). Unless $\Delta t \to 0$, equation (19) shows that the CRR model in (17) is not consistent with the underlying lognormal distribution.

$$
\begin{aligned}
qu + (1-q)d &= e^{r\Delta t}. & (18)\\
u^2 q + d^2(1-q) - (qu + (1-q)d)^2 &= (u+d)e^{r\Delta t} - 1 - e^{2r\Delta t} \\
&= (e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{r\Delta t} - 1 - e^{2r\Delta t}. & (19)
\end{aligned}
$$

Clearly, the mean is matched but the variance is not. Note that with q as in (17), any value of u and d would satisfy (18). The required modification is therefore to use risk-neutral probabilities in (17) and match the second central moment of the discrete distribution to the variance of the continuous distribution in (A5). This gives the correct u and d. That is,

$$
\begin{aligned}
& u^2 q + d^2(1-q) - (qu + (1-q)d)^2 = (e^{(2r+\sigma^2)\Delta t} - e^{2r\Delta t}), \\
\Leftrightarrow \quad & q(1-q)(u-d)^2 = (e^{(2r+\sigma^2)\Delta t} - e^{2r\Delta t}), \\
\Leftrightarrow \quad & (e^{r\Delta t} - d)(u - e^{r\Delta t}) = (e^{(2r+\sigma^2)\Delta t} - e^{2r\Delta t}), \\
\Leftrightarrow \quad & u = \frac{e^{(2r+\sigma^2)\Delta t} - de^{r\Delta t}}{e^{r\Delta t} - d}.
\end{aligned}
$$

Let

$$ud = \lambda. \tag{20}$$

Then,

$$u = \frac{e^{-r\Delta t}}{2}\left(\lambda + e^{(2r+\sigma^2)\Delta t} + \sqrt{\left(\lambda + e^{(2r+\sigma^2)\Delta t}\right)^2 - 4\lambda e^{2r\Delta t}}\right) \tag{21}$$

and

$$d = \frac{e^{-r\Delta t}}{2}\left(\lambda + e^{(2r+\sigma^2)\Delta t} - \sqrt{\left(\lambda + e^{(2r+\sigma^2)\Delta t}\right)^2 - 4\lambda e^{2r\Delta t}}\right). \tag{22}$$

Equations (20)-(22) represent jump amplitudes for the modified CRR (MCRR) binomial model. The most common specification is to have $\lambda = 1$. In fact, any value of λ sufficiently close to one yields good results. For example, the choice $\lambda = e^{r\Delta t}$ seems natural.

Comparison of Numerical Methods

Horasanli (2007), Kamrad and Ritchken (1991) discussed the rates of convergence for both trinomial and binomial models, and noted that for the same number of iterations, the former yield small errors (with respect to the Black and Scholes model) but are computationally more expensive.
Figure 1 shows errors between the true Black and Scholes and computed European option prices using binomial and trinomial models for different iterations. The trinomial model used here is one described by

[5]Using Taylor series expansion, it can be shown that probability q in (17) is equivalent to probability $\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}$ in (16).

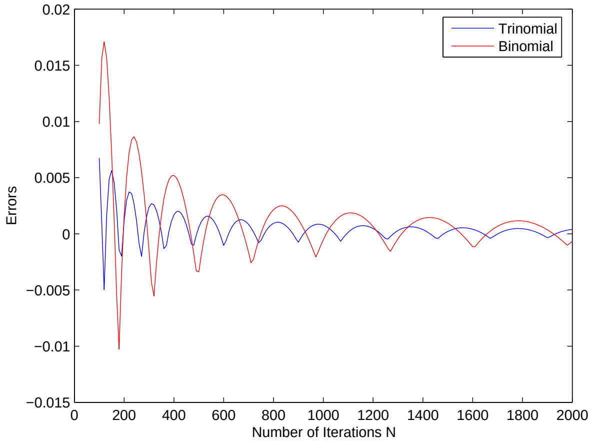

Figure 1: *Convergence rates of Trinomial and Binomial models for a European put option. The number of time points* N *increases from* 50 *to* 2000. $S_0 = 100, r = 0.06, E = 110, \sigma = 0.3, q_2 = 1/3$ *and* $T = 0.5$.

equations (5) and (10) (with $q_2 = \frac{1}{3}$), while the binomial model is the MCRR (with $\lambda = 1$). The current stock price $S_0 = 100$, the strike price $E = 110$, time remaining to maturity $T = 6$ months, the riskfree rate $r = 6\%$ and the volatility $\sigma = 30\%$. Clearly, the amplitudes (representing errors) in the binomial model are bigger than the amplitudes in the trinomial model.

A finite difference approach (trinomial model) for pricing lookback options is included in the appendix. This model can be viewed as an extension of the single state variable binomial model for pricing lookback options which was proposed by Cheuk and Vorst (1997). Dai (2000) showed that the speed of convergence for the model of Cheuk and Vorst (1997) was low due to a poor approximation of the Neumann condition. A better speed of convergence can be attained if one used a trinomial model of Kamrad and Ritchken (1991) with the right boundary conditions.

FDMM and MCRR binomial model are compared with the CRR model. First and foremost, the CRR model is consistent with the continuous model only in the limit (as $\Delta t \to 0$), whereas the MCRR model is consistent with the continuous model for any time step. Moreover, it is known (See for example Trigeorgis, 1991, p.319), (Jabbour et al., 2001, p.992) that the CRR model can give rise to negative probabilities, and is hence unstable.

In Figure 2, CRR, MCRR (with $\lambda = 1$) and FDMM binomial methods are compared by plotting values for a European put option at the money for an increasing number of time points N and for different maturities. The dotted graph represents Black and Scholes values. Figure 3 shows absolute errors with respect to the Black and Scholes values. The KEY to plots in Figure 2 is given in Figure 3.

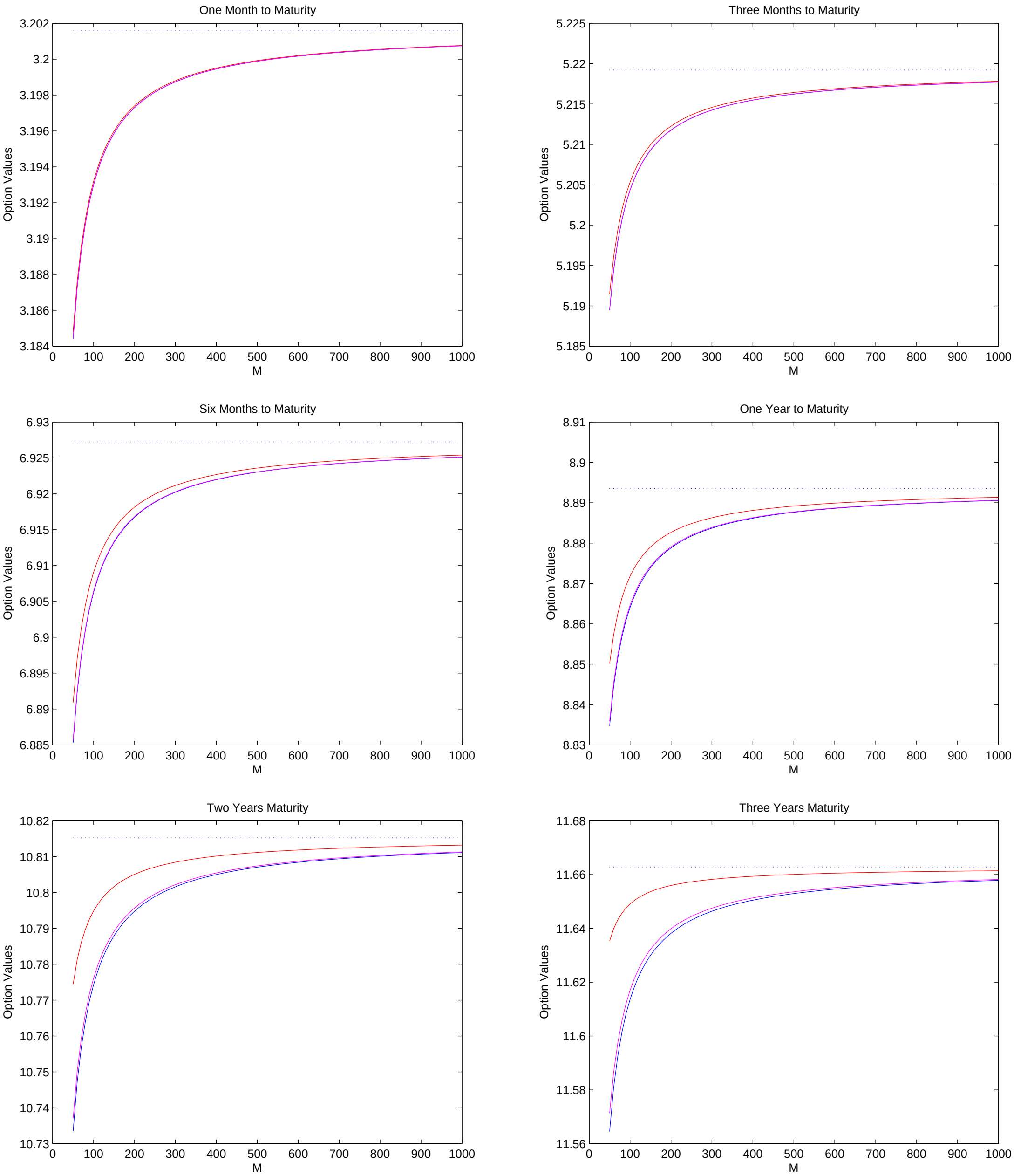

Figure 2: Convergence of binomial models for a European put at the money. The number of time points N increases from 50 to 1000. $S_0 = 100, r = 0.06, \sigma = 0.3$.

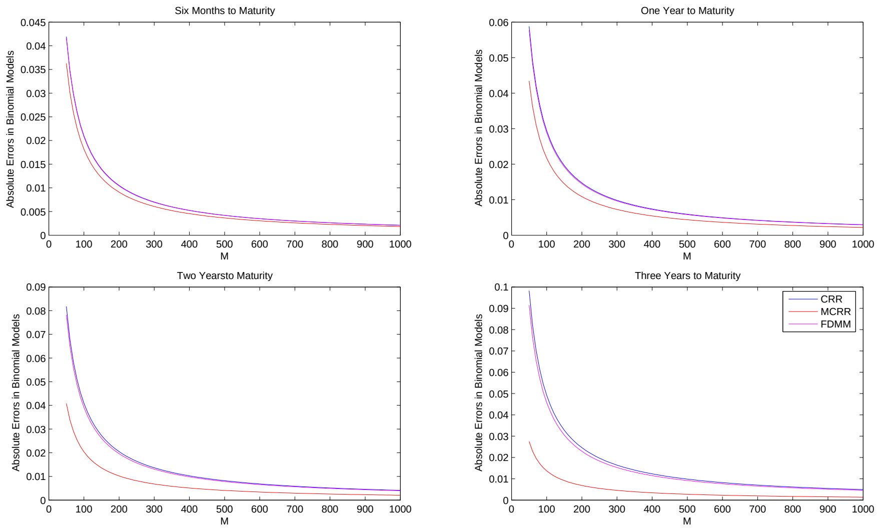

Figure 3: Error in the binomial models for a European Put option at the money for varying maturities. The number of time points N increases from 50 to 1000. $S_0 = 100, \sigma = 0.3, r = 0.06$.

Observations

Put option values derived from the three binomial models do coincide for short lived options at the money. This is also true for short lived options in the money and out of the money (Figure 4). For longer maturities (more than one year), the FDMM options values coincide with CRR option values (Figures 2 and 4). The MCRR option values are slightly better than the CRR and FDMM for long lived options at the money (Figures 2 and 3). This advantage disappears for out of the money and in the money options (Figure 4).

The drawback of many binomial models is that the convergence to the continuous model of Black and Scholes is non-monotonic. The FDMM and MCRR are no better in this respect. Tian (1999) developed a flexible binomial (FB) model with a 'tilt' parameter that alters the shape and span of a binomial lattice. By recalibrating the binomial model through the tilt parameter, an improvement in the convergence of CRR model is made. For the put option at the money, FDMM and MCRR binomial models perform as good as the FB model.

Conclusion

An explicit finite difference scheme was used to determine jump probabilities for a log-transformed trinomial lattice and a moment matching technique was then used to determine the corresponding jump amplitudes. For computational efficiency, some restrictions were made on the middle jump amplitude and as a result trinomial models in Kamrad and Ritchken (1991); Horasanli (2007) and Chen et al. (2002) were derived. In addition, a new parametrization for the binomial option pricing model was given.

The famous CRR binomial option pricing model was re-examined and alternative jump amplitudes were given. Unlike the CRR model which is only consistent in the limit as the size of time steps tends to zero, the modified model is consistent with the underlying continuous distribution at all time steps. The binomial models proposed converge to the continuous limit in the same way the CRR model converges to the continuous model of Black and Scholes. In particular, the convergence is non-monotonic. One noted advantage though is that, for the put options at the money, the MCRR model exhibit slightly smaller errors with respect to the true values.

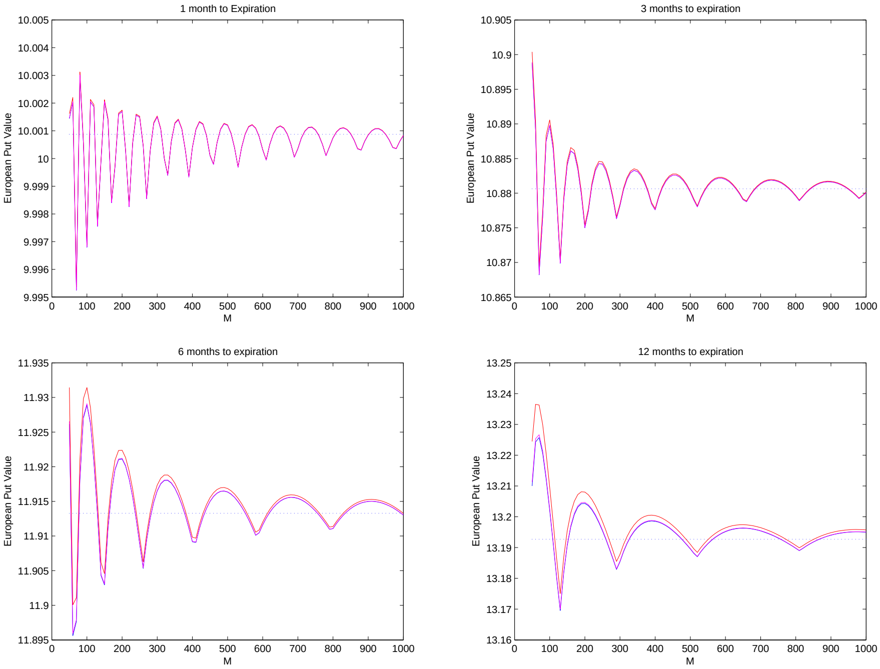

Figure 4: Convergence of binomial models for a European Put as the number of time points N increases from 50 to 1000. $S_0 = 90$, Strike price $E = 100, r = 0.06, \sigma = 0.3$.

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References

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Political Economy*, 81:637–654.

Boyle, P. (1986). Option valuation using a three jump process. *International Options Journal*, 3:7–12.

Boyle, P. (1988). A lattice framework for option pricing with two state variables. *Financial and Quantitative Analysis*, 23:1–12.

Boyle, P., Evnine, J., and Gibbs, S. (1989). Numerical evaluation of multivariate contingent claims. *Review of Financial Studies.*, 2:241–250.

Brennan, M. and Schwartz, E. (1977). The valuation of american put options. *Finance*, 32:449–462.

Brennan, M. and Schwartz, E. (1978). Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. *Financial and Quantitative Analysis*, 13:461–474.

Chen, H.-C., Chen, M., D., and Chung, S.-L. (2002). The accuracy and efficiency of alternative option pricing approaches relative to a log-transformed trinomial model. *Futures Markets*, 22:557–577.

Cheuk, H., F. T. and Vorst, C., F. T. (1997). Currency lookback options and observation frequency: A binomial approach. *International Money and Finance*, 16:173–187.

Cox, John, C., Ross, Stephen, A., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Financial Economics*, 7:229–263.

Dai, M. (2000). A modified binomial tree method for currency lookback options. *Acta Mathematica Sinica, English Series*, 16:445–454.

He, H. (1990). Convergence from discrete- to continuous-time contingent claims prices. *The Review of Financial Studies*, 3:523–546.

Horasanli, M. (2007). A comparison of lattice based option pricing models on the rate of convergence. *Applied Mathematics and Computation*, 184:649–658.

Jabbour, M., G., Kramin, V., M., Kramin, V., T., and Young, D., S. (2005). Multinomial lattices and derivative pricing. *Advances in Quantitative Analysis of Finance and accounting-New Series.*, 2:1–16.

Jabbour, M., G., Kramin, V., M., and Young, D., S. (2001). Two-state option pricing:binomial models revisited. *Futures Markets*, 21:987–1001.

Jarrow, R. and Rudd, A. (1983). *Option Pricing*. Irwin Professional Pub.

Kamrad, B. and Ritchken, P. (1991). Multinomial approximating models for options with k state variables. *Management Science*, 37:1640–1652.

Omberg, E. (1988). Efficient discrete time jump process models in option pricing. *Journal of Financial and Quantitative Analysis*, 23:161–174.

Rendleman, J., R. and Bartter, J., B. (1979). Two-state option pricing. *Finance*, 34:1093–1110.

Tian, Y., S. (1993). A modified lattice approach to option pricing. *Futures Markets*, 13:563–577.

Tian, Y., S. (1999). A flexible binomial option pricing model. *Futures Markets*, 19:817–843.

Trigeorgis, L. (1991). A log-transformed binomial numerical analysis method for valuing complex multi-option investments. *Financial and Quantitative Analysis*, 26:309–326.

Appendix

Mean and Variance of a Geometric Brownian Motion

Consider a stochastic process S that follows (in a risk-neutral world) an Ito process

$$dS = rSdt + \sigma SdW, \tag{A1}$$

where W is a Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The instantaneous risk-free rate r and volatility σ of the underlying stochastic variable are considered to be deterministic constants.

Letting

$$X = \ln S \tag{A2}$$

and using Ito's Lemma gives

$$dX = (r - \frac{\sigma^2}{2})dt + \sigma dW,$$

whose solution over any time interval $(t, t + \Delta t)$ is

$$X_{t+\Delta t} - X_t = (\alpha - \frac{\sigma^2}{2})\Delta t + \sigma \Delta W;\ \Delta W = W_{t+\Delta t} - W_t \sim N(0, \Delta t). \tag{A3}$$

From equation (A3) one can deduce that,

$$S_{t+\Delta t} = S_t e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma \Delta W}, \tag{A4}$$

where S_t is known at time t.

The moment generating function $M_{\Delta W}(.)$ of ΔW is $e^{\frac{\Delta t(.)}{2}}$. Therefore,

$$E(S_{t+\Delta t}) \quad = \quad S_t e^{(r-\frac{\sigma^2}{2})\Delta t} M_{\Delta W}(\sigma) = S_t e^{r\Delta t}$$

and

$$E(S^2_{t+\Delta t}) \quad = \quad S_t^2 e^{2(r-\frac{\sigma^2}{2})\Delta t} M_{\Delta W}(2\sigma) = S_t^2 e^{(2r+\sigma^2)\Delta t}.$$

The variance of the return process $\frac{S_{t+\Delta t}}{S_t}$ is given as

$$Var\left(\frac{S_{t+\Delta t}}{S_t}\right) \quad = \quad E\left(\frac{S_{t+\Delta t}}{S_t}\right)^2 - \left(E\frac{S_{t+\Delta t}}{S_t}\right)^2 = (e^{(2r+\sigma^2)\Delta t} - e^{2r\Delta t}). \tag{A5}$$

Pricing of Lookback Options

Let S_t be the exchange rate between the domestic currency and the foreign currency and let r_d and r_f be the domestic and foreign short rates respectively, and are assumed to be deterministic constants.

The maximum M_t and the minimum processes m_t of S_t are defined as

$$M_t \quad = \quad \max_{0\le u\le t} S_u,\ m_t = \min_{0\le u\le t} S_u.$$

We want to determine a fair price of a floating strike lookback call option.[6]

The Black and Scholes PDE for the price of a European currency lookback call option is given by

$$\begin{cases} \frac{\partial C}{\partial t} + S(r_d - r_f)\frac{\partial C}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C}{\partial S^2} - r_d C = 0, 0 \le m \le S, 0 \le t \le T, \\ C(T, S, m) = S - m,\ \frac{\partial C}{\partial m}(t, m, m) = 0. \end{cases} \tag{B1}$$

[6]For a call, the terminal payoff is $S_T - m_T$ and in case of a put, the terminal payoff is $M_T - S_T$.

Using the transformation $x = \ln(\frac{S}{m}), SV(t,x) = C(t,m,S)$ and Ito's lemma, equation (B1) can be transformed into

$$\begin{cases} \frac{\partial V}{\partial t} + \left(r_d - r_f + \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - r_f V = 0, x > 0, 0 \le t \le T \\ V(T,x) = 1 - e^{-x},\ \frac{\partial V}{\partial x}(t,0) = 0. \end{cases} \tag{B2}$$

Similarly, as in (3), a discretization of equation (B2) yields the following explicit difference scheme:

$$\begin{aligned} V_j^n &= \frac{1}{1+r_f\Delta t}\left\{\left(\frac{\sigma^2\Delta t}{2\Delta x^2} + \frac{\Delta t\left(r_d - r_f + \frac{\sigma^2}{2}\right)}{2\Delta x}\right)V_{j+1}^{n+1} + \left(1 - \frac{\sigma^2\Delta_t}{\Delta x^2}\right)V_j^{n+1}\right. \\ &\left. + \left(\frac{\sigma^2\Delta t}{2\Delta x^2} - \frac{\Delta t\left(r_d - r_f + \frac{\sigma^2}{2}\right)}{2\Delta x}\right)V_{j-1}^{n+1}\right\}, j \ge 1,\ 0 \le n \le N-1. \end{aligned}$$

The discretization of the Neumann boundary condition $\frac{\partial V}{\partial x}(t,0) = 0$ gives

$$V_{-1}^{n+1} = V_1^{n+1}. \tag{B3}$$

Therefore,

$$V_0^n = \frac{1}{1+r_f\Delta t}\left\{\left(\frac{\sigma^2\Delta t}{\Delta x^2}\right)V_1^{n+1} + \left(1 - \frac{\sigma^2\Delta t}{\Delta x^2}\right)V_0^{n+1}\right\},\ 0 \le n \le N-1; \tag{B4}$$

and the terminal condition is

$$V_j^N = 1 - e^{-j\Delta x}, j \ge 0. \tag{B5}$$

Define $\Delta x = \lambda\sigma\sqrt{\Delta t},\ \ a_1 = e^{\Delta x},\ \ a_3 = e^{-\Delta x}$. Then, equations (B3)-(B5) can be re-arranged into the following single equation:

$$\begin{cases} V_j^n = \frac{1}{1+r_f\Delta t}\left\{\hat{q}_1 V_{j+1}^{n+1} + \hat{q}_2 V_j^{n+1} + \hat{q}_3 V_{j-1}^{n+1}\right\};\ \ j \ge 1, 0 \le n \le N-1, \\ V_0^n = \frac{1}{1+r_f\Delta t}\left\{(1-\hat{q}_2)V_1^{n+1} + \hat{q}_2 V_0^{n+1}\right\}; 0 \le n \le N-1, \\ V_j^N = 1 - a_3^j,\ \ 0 \le j \le N; \end{cases} \tag{B6}$$

where

$$\hat{q}_{1,3} = \frac{1}{2\lambda^2} \pm \frac{\sqrt{\Delta t}\left(r_d - r_f + \frac{\sigma^2}{2}\right)}{2\lambda\sigma},\ q_2 = 1 - q_1 - q_3.$$

Once again we recognize risk neutral probabilities q_1, q_2 and q_3 on imposing the nonnegativity condition similar to (6).

The (trinomial) model in (B6) can be viewed as an extension of a single state variable binomial model proposed by Cheuk and Vorst (1997) for pricing lookback options. In their model, Cheuk and Vorst set $\Delta x = \sigma\sqrt{\Delta t}$ and then approximated the Neumann boundary condition by

$$V_{-1}^{n+1} = V_0^{n+1} \tag{B7}$$

instead of (B3). This way, they were able to approximate the continuous evaluation model with the binomial model. Ideally, the model developed by Cheuk and Vorst (1997) can be viewed as the binomial model of Cox et al. (1979) applied to pricing of lookback options while the model in (B6) can be viewed as the trinomial model of Kamrad and Ritchken (1991) applied to pricing of lookback options. Dai (2000) noted that the model of Cheuk and Vorst was very slow in convergence and this was attributed to the poor approximation in (B7). Dai then introduced a correction term and was able to obtain better results. Since (B3) is the 'true' approximation of the Neumann condition, the convergence of trinomial results will be much better than the binomial model and also better than the improved version of Dai (2000).